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# AXISYMMETRICAL VIBRATIONS OF SHELLS OF REVOLUTION UNDER ABRUPT LOADING†

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A frequency method is proposed for solving the problem of the vibrations of shells of revolution taking into account the energy dissipation under arbitrary force loading and on collision with a rigid obstacle. The Laplace transform is taken of the equation of the vibrations of a shell of revolution with non-zero initial conditions. For the inhomogeneous differential equation obtained, a variational method is used to solve the boundary-value problem, which consists of finding the Laplace-transformed boundary transverse and longitudinal forces and bending moments as functions of the boundary displacements. The equations of equilibrium of nodes, i.e. the corresponding equations of the finite-element method, are then compared, using results obtained earlier [1–4]. Amplitude–phase–frequency characteristics (APFCs) for the shell cross-sections selected are plotted. An inverse Laplace transformation is carried out using the clear relationship between the extreme points of the APFCs and the coefficients of the corresponding terms of the series in an expansion vibration modes [3]. In view of the fact that the proposed approach is approximate, numerical testing is used. © 2002 Elsevier Science Ltd. All rights reserved.

The problem of the longitudinal vibrations of elastic rods of variable stepped cross-section on collision with a rigid obstacle was solved by a similar method in [5]. Below, unlike the procedure set out earlier in [5], the coefficients of the mass matrix and stiffness matrix are obtained from variational considerations.

The equations of the dynamics of a linear viscoelastic system in operator form can be written as follows:

$$D\sigma + R\frac{\partial^2 u}{\partial t^2} + T\frac{\partial u}{\partial t} - f = 0, \quad CD^* u + C_1 D^* \frac{\partial u}{\partial t} = \sigma \tag{1}$$

where $\sigma$ is the vector of generalized forces or the stress tensor, $u$ is the vector of generalized displacements, $R$ is the matrix of inertia characteristics or the specific mass, $T$ is the matrix of external energy dissipation, $f$ is the vector function of external loads, and $C$ and $C_1$ are respectively the matrices or tensors of the constants of elasticity and coefficients of internal friction.

We will take the boundary conditions in the form

$$n_\sigma \sigma = f_S \ \text{ on } \ S_1, \quad n_u u = u_S \ \text{ on } \ S_2 \tag{2}$$

where $n_\sigma$ and $n_u$ are the corresponding operators of static and geometric compatibility on the body surface, $f_S$ are the loads on the surface area $S_1$ and $u_S$ are the boundary displacements on $S_2$.

We will write the compatibility conditions on the boundaries of the finite elements

$$n_{\sigma+}\sigma_+ + n_{\sigma-}\sigma_- = 0 \ \text{ on } \ S_1', \quad n_{u+}u_+ = n_{u-}u_- \ \text{ on } \ S_2 \tag{3}$$

Here, the plus and minus subscripts correspond to different sides of the interface of the elements $S' = S_1' \cup S_2'$.

We will take the initial conditions in the form

$$t = 0: u = a_0, \ \partial u / \partial t = a_1 \tag{4}$$

where $a_0$ and $a_1$ are respectively the fields of the initial displacements and initial velocities.

The operators $D$ and $D^*$ are conjugate in the Lagrangian sense, i.e.

$$\int_V (D\sigma)^T u dV = \int_V \sigma^T D^* u dV - \int_S \sigma_S^T u_S dS \tag{5}$$

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where $\sigma_S = n_\sigma \sigma$, $u_S = n_u u$ and $V$ is the volume of a finite element. In the general case, the boundary of an element $S = S_1 \cup S_2 \cup S'_1 \cup S'_2$.

Operator equations (1), boundary conditions (2) and compatibility conditions (3) are valid for rods, plates and shells. The methods discussed here are therefore universal for all applied problems of linear viscoelasticity [6].

We will take the Laplace transform of Eqs (1), boundary conditions (2) and compatibility conditions (3)

$$D\sigma + R(p^2 - pa_0 - a_1) + T(pu - a_0) - f = 0, \quad (C + C_1 p)D^* u - C_1 D^* a_0 = \sigma \tag{6}$$

$$n_\sigma \sigma = f_S \text{ on } S_1, \quad n_u u = u_S \text{ on } S_2 \tag{7}$$

$$n_{\sigma+}\sigma_+ + n_{\sigma-}\sigma_- = n_\sigma \sigma' \text{ on } S'_1, \quad n_{u+}u_+ - n_{u-}u_- = n_u u' = 0 \text{ on } S'_2 \tag{8}$$

where

$$u = u(p), \quad u(p) = \int_0^\infty u(t) e^{-pt} dt; \quad \sigma = \sigma(p), \quad \sigma(p) = \int_0^\infty \sigma(t) e^{-pt} dt$$

The following theorem [3], extending the results of [1, 2] to problems of viscoelasticity, holds: the Laplace-transformed equations (6), boundary conditions (7) and compatibility conditions (8) for the generalized displacements and generalized forces of a viscoelastic body are equivalent to the condition for the following functional to be stationary:

$$\begin{aligned} e(p) = {} & \frac{1}{2}\int_V [D\sigma + p^2 Ru + pTu - 2(f + pRa_0 + Ra_1 + Ta_0)]^T u dV + \\ & + \frac{1}{2}\int_V \sigma^T (D^* u - C^{*-1}\sigma - 2C^{*-1} C_1 D^* a_0) dV + \frac{1}{2}\int_{S_1} (n_\sigma \sigma - 2f_S)^T n_u u dS_1 - \\ & - \frac{1}{2}\int_{S_2} (n_\sigma \sigma)^T (n_u u - 2u_S) dS_2 + \frac{1}{2}\int_{S'_1} (n_\sigma \sigma')^T n_u u dS'_1 - \frac{1}{2}\int_{S'_2} (n_\sigma \sigma)^T n_u u' dS'_2 \end{aligned} \tag{9}$$

where $C^* = C + C_1 p$ and $V$ is the volume of the elements into which the body is divided. Here, the sign of summation over the elements is omitted.

We will examine a single field of displacements. Following the variational method, we will seek a solution in the form

$$u = \sum_j \mu_j u_j, \quad \sigma = \sum_j \mu_j C^* D^* u_j + C_1 D^* a_0 \tag{10}$$

Variations in $u$ and $\sigma$ will have the form

$$\delta u = \sum_j \delta\mu_j u_j, \quad \delta\sigma = \sum_j \delta\mu_j C^* D^* u_j, \quad \delta u(p) = \int_0^\infty \delta u(t) e^{-pt} dt \tag{11}$$

where $u_j$ are the corresponding coordinate functions satisfying the compatibility conditions on the boundary of an element, and variation in $u$ is understood in the sense indicated above.

By varying functional (9), having satisfied the second of Eqs (6), i.e. the law of viscoelasticity, and satisfying the strain compatibility conditions on the boundary between the elements, taking into account relations (5), (10) and (11), we obtain

$$\begin{aligned} & \int_V \{(C^* D^* u - C_1 D^* a_0)^T + [p^2 Ru + pTu - (f + pRa_0 + Ra_1 + Ta_0)]^T\} u_j dV - \\ & - \int_{S_1} f_S^T n_u u_j dS_1 = 0, \quad j = 1, \ldots, l \end{aligned} \tag{12}$$

where $l$ is the number of degrees of freedom of the finite element.

Equation (12) is a generalized form of the equations of the finite-element method based on nodal displacements. The number of such equations is equal to the number of nodal displacements or, in other words, to the number of degrees of freedom $N$ of the discrete model.

From Eq. (12) we obtain the corresponding expressions for elements of the stiffness, energy dissipation, mass and loading term matrices

$$
\begin{aligned}
&C_{ij} = \int_V (CD^* u_i)^T D^* u_j dV, \quad C_{1ij} = \int_V (C_1 D^* u_i)^T D^* u_j dV, \quad T_{ij} = \int_V (Tu_i)^T u_j dV \\
&R_{ij} = \int_V (Ru_i)^T u_j dV, \quad f_j = \int_V (f + pRa_0 + Ra_1 + Ta_0)^T u_j dV + \\
&+\int_V (CD^* a_0)^T D^* u_j dV + \int_{S_1} f_S^T u_j dS_1 = f_j(p) + f_{1j} + f_{2j} p
\end{aligned}
\tag{13}
$$

where $C$ is the matrix of the constants of elasticity

$$
C = \begin{Vmatrix} B_1 & 0 \\ 0 & B_2 \end{Vmatrix}, \quad B_k = \begin{Vmatrix} b_k & b_k\mu & 0 \\ b_k\mu & b_k & 0 \\ 0 & 0 & b_k \dfrac{1-\mu}{2} \end{Vmatrix}, \quad k = 1, 2
$$

$$
b_1 = \frac{Eh}{1-\mu^2}, \quad b_2 = \frac{Eh^3}{12(1-\mu^2)}
$$

$b_1$ is the stiffness for tension, $b_2$ is the cylindrical stiffness for bending, $E$ is the modulus of elasticity, $\mu$ is Poisson's ratio and $h$ is the shell thickness. In the case of a shell of revolution, the form of the operators $D$ and $D^*$ follows from equations given earlier [7, pp. 33 and 39].

The operators $D$ and $D^*$ possess the property

$$
\begin{aligned}
&\int_\Sigma [(Dx)^T y - x^T D^* y] d\Sigma = \\
&= -\left\{ \int_0^{2\pi} \left[ T_1 u + M_1 \gamma_1 + \left( S + \frac{2H}{r_2} \right) v + \frac{1}{\upsilon} \left( \frac{\partial \upsilon M_1}{\partial S} - M_2 \cos\theta + 2\frac{\partial H}{\partial \varphi} \right) w \right]_{\alpha_n}^{\alpha_k} \upsilon d\varphi - \right. \\
&\left. - \int_{\alpha_0}^{\alpha_1} \left( Su + \frac{2Hu}{\upsilon r_1} + \frac{T_2 v}{\upsilon} + \frac{M_2 \sin\theta}{\upsilon^2} v - 2H \frac{r_1 \cos\theta}{\upsilon^2} w \right) \Bigg|_0^{2\pi} d\alpha \right\} = \\
&= -\int_0^{2\pi} \left[ T_1 u + \left( S + \frac{2H}{r_2} \right) \upsilon + \left( N_1 + \frac{\partial H}{\partial \varphi} \right) w - M_1 \gamma_1 \right]_{\alpha_n}^{\alpha_k} \upsilon d\varphi = -\int_\Gamma x_\Gamma^T y_\Gamma d\Gamma_1 \\
&x_\Gamma^T = \left( T_1, S + \frac{2H}{r_2}, N_1 + \frac{\partial H}{\partial \varphi}, -M_1 \right), \quad y_\Gamma^T = (u, v, w, \gamma_1)
\end{aligned}
\tag{14}
$$

where $\Sigma$ is the shell surface, $x^T = (T_1, T_2, S, M_1, M_2, H)$ is the vector of the forces, $T_1, M_1$ and $T_2, M_2$ are the meridional and peripheral tensile forces and bending moments, $S = S_{12} - H_{21}/r_2 = S_{21} - H_{12}/r_1$, $H = (H_{12} + H_{21})/2$, $S_{12}$, $S_{21}$, $H_{12}$ and $H_{21}$ are shear forces and torques, $N_1$ is the transverse force per unit length of the parallel, $y^T = (u, \upsilon, w)$ is the vector of displacements, $u$ is the displacement along the tangent to the meridian, $\upsilon$ is the displacement along the tangent to the parallel, $w$ is the normal displacement, $x_\Gamma^T$ and $y_\Gamma^T$ are respectively the vectors of generalized forces and displacements on the edges of the element, $r_1$ and $r_2$ are the principal radii of curvature of the shell, $\upsilon$ is the radius of curvature of the parallel, $\alpha_n$ and $\alpha_k$ are respectively the initial and final values of the meridional arc coordinate of an isolated element with the surface $\Sigma$ and $\alpha$ is an arc meridional coordinate. When evaluating integral (14), account is taken of the periodicity with respect to angle $\theta$.

Equations (1) must be supplemented with the corresponding boundary conditions, which follow from the properties of the operators (14),

$$
x_\Gamma \big|_{\Gamma_1} = f_\Gamma, \quad y_\Gamma \big|_{\Gamma_2} = u_\Gamma \tag{15}
$$

where $\Gamma_1$ is the part of the contour where the forces are specified, and $\Gamma_2$ is the part of the contour where displacements are specified. Division of the boundary contour into $\Gamma_1$ and $\Gamma_2$ is considered to

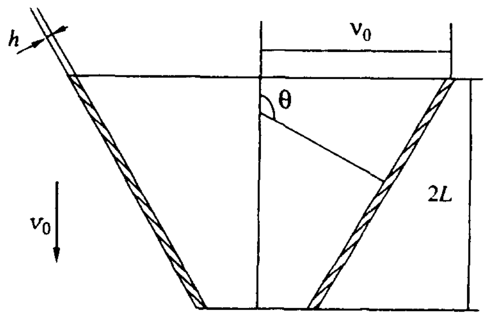


Fig. 1

be nominal, since individual components of the vector of generalized forces and additional components of the vector of displacements can be specified on the same section.

For a conical shell (Fig. 1), where $r_1 = \infty$, under an axisymmetrical load, the quantity related to displacement υ is not considered, and the operator $D^*$ and the matrix $C$ are transformed into

$$D^* = \begin{Vmatrix} d/d\alpha & 0 \\ \upsilon^{-1}\cos\theta & \upsilon^{-1}\sin\theta \\ 0 & -d^2/d\alpha^2 \\ 0 & -\upsilon^{-1}\cos\theta d/d\alpha \end{Vmatrix}, \quad C = \begin{Vmatrix} b_1 & \mu b_1 & 0 & 0 \\ \mu b_1 & b_1 & 0 & 0 \\ 0 & 0 & b_2 & \mu b_2 \\ 0 & 0 & \mu b_2 & b_2 \end{Vmatrix}$$

We will assume a field of displacements in the form

$$u = \beta_1 + \beta_2\alpha, \quad w = \beta_3 + \beta_4\alpha + \beta_5\alpha^2 + \beta_6\alpha^3$$

where $\beta_1, \ldots, \beta_6$ are unknown coefficients. This concept may be justified by the fact that the equation of bending is only slightly related to the equation of extension [4].

We will find the form functions. They are defined by the relation [4]

$$\begin{Vmatrix} u \\ w \end{Vmatrix} = \begin{Vmatrix} U_1 & 0 & 0 & U_4 & 0 & 0 \\ 0 & U_2 & U_3 & 0 & U_5 & U_6 \end{Vmatrix} (u_i\ w_i\ \gamma_i\ u_j\ w_j\ \gamma_j)^T$$

where $U_1, \ldots, U_6$ are the form functions, and $u_i$, $w_i$, $\gamma_i$, $u_j$, $w_j$ and $\gamma_j$ are the displacements and angles of rotation of the boundary cross-sections.

We will introduce the variable $\alpha/L = \alpha_1$, where $L$ is the length of the element along the generatrix. Then

$$U_1 = 1 - \alpha_1, \quad U_2 = 1 - 3\alpha_1^2 + 2\alpha_1^3, \quad U_3 = L(\alpha_1 - 2\alpha_1^2 + \alpha_1^3)$$

$$U_4 = \alpha_1, \quad U_5 = 3\alpha_1^2 - 2\alpha_1^3, \quad U_6 = L(-\alpha_1^2 + \alpha_1^3)$$

Changing from a local system of axes $(\overline{\tau}_1, \overline{\tau}_2, \overline{n})$ to a global system $(X, Y, Z)$, we will express the vector y in terms of the matrix of the form functions $U$ and the nodal displacements

$$z^T = (u_i, w_i, dw_i/d\alpha, \ u_j, w_j, dw_j/d\alpha)$$

From formulae (11), taking into account that $V$ corresponds to $\Sigma$, and $d\Sigma = 2\pi\nu L d\alpha_1$, we obtain, for the stiffness and mass matrices, the expressions [8]

$$\overline{C} = 2\pi L\Lambda^T\left\{\int_0^1 (D^*U)^T CD^*U\nu d\alpha_1\right\}\Lambda, \quad \overline{R} = 2\pi L\Lambda^T\left\{\int_0^1 (RU)^T U\nu d\alpha_1\right\}\Lambda$$

$$\overline{C} = \Lambda^T C_0 \Lambda, \quad \overline{R} = \Lambda^T R_0 \Lambda, \quad i, j = 1, \ldots, 6$$

$$\Lambda = \begin{Vmatrix} M & 0 \\ 0 & M \end{Vmatrix}, \quad M = \begin{Vmatrix} \sin\theta & -\cos\theta & 0 \\ \cos\theta & -\sin\theta & 0 \\ 0 & 0 & 1 \end{Vmatrix}$$

To set up a system of equations, we will represent the stiffness matrix and the mass matrix divided into 3 × 3 blocks

$$C_0 = \begin{Vmatrix} A_{ij} & B_{ij} \\ B_{ji} & A_{ji} \end{Vmatrix}, \quad R_0 = \begin{Vmatrix} m_{ij}^A & m_{ij}^B \\ m_{ji}^B & m_{ji}^A \end{Vmatrix}$$

while the load vector $f^T = (f_{ij}, f_{ji})$. Then, for a shell we obtain the following recurrence relation

$$(B_{g-1g} - \omega^2 m_{g-1g}^B)Y_{g-1} + [A_{g-1g} + A_{g+1g} - \omega^2(m_{g-1g}^A + m_{gg+1}^A)]Y_g +$$
$$+ (B_{gg+1} - \omega^2 m_{gg+1}^B)Y_{g+1} = f_{g-1g} + f_{gg+1} + f_g \quad (16)$$

where $g$ is the number of the characteristic cross-section, $f_g$ is the load vector in cross-section $g$, and $Y_g$ and $Y_{g\pm1}$ are the vectors of nodal displacements.

According to Eq. (12), to take into account the energy dissipation, we will substitute for $E$ the quantity $E(1 + i\omega\gamma)$, where $\gamma$ is the coefficient of internal energy dissipation, and, by solving system of equations (12), we plot the APFCs. A mathematical model of the shell of revolution is formed from the characteristic points of the APFCs in the form

$$W(\omega) = \sum_j \frac{k_j}{-T_{2j}^2\omega^2 + T_{1j}\omega i + 1} \quad (17)$$

$$k_j = A_j \frac{T_{1j}}{T_{2j}}, \quad T_{2j} = \frac{1}{\omega_{1j}}, \quad T_{1j} = T_{2j}\left(1 - \left(\frac{\omega_{2j}}{\omega_{1j}}\right)^2\right)$$

where $A_j$ is the vertical dimension of the APFC loop.

*Examples of a numerical test.* The following problem is examined: at a certain instant of time, a force $q_1 = 10$ kN/m$^2$ begins to act on a conical shell (Fig. 1) with parameters $\theta = 45°$, $L = 6$ m, $\nu_0 = 20$ m, $h = 0.005$ m, $E = 2.1 \times 10^{11}$ N/m$^2$, $\mu = 0.3$ and $q = 100$ N/m$^2$; it is required to plot the transient with a single pulse and a stepped disturbance corresponding to the pressure indicated above.

Solving Eqs (16), we plot the APFCs. Then, from formulae (17), we plot the transfer functions for the selected cross-sections of the shell. For the middle cross-section of the shell, in expansion (17), eight significant terms are obtained. The coefficients of these terms are given below

| $j$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| $-k_j \times 10^{12}$, m$^3$/N | 12070 | −1946 | −9212 | 7828 | 5879 | 1277 | 1865 | 4444 |
| $T_{1j} \times 10^9$, s | 6531 | 5864 | 5165 | 4670 | 4221 | 3773 | 3519 | 2168 |
| $T_{2j}^2 \times 10^7$, s$^2$ | 6478 | 5846 | 5193 | 4758 | 4289 | 3921 | 3562 | 2164 |

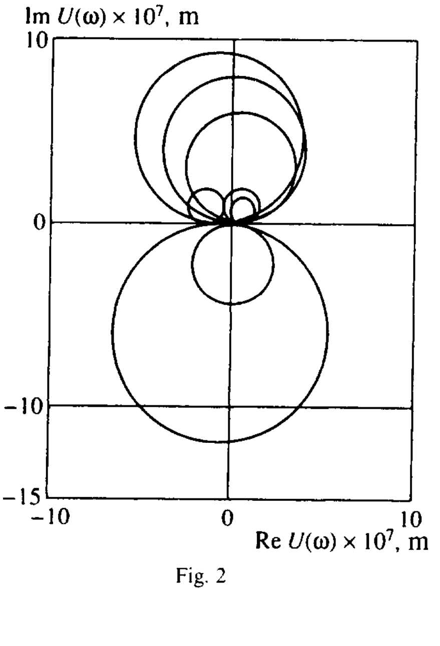


Fig. 2

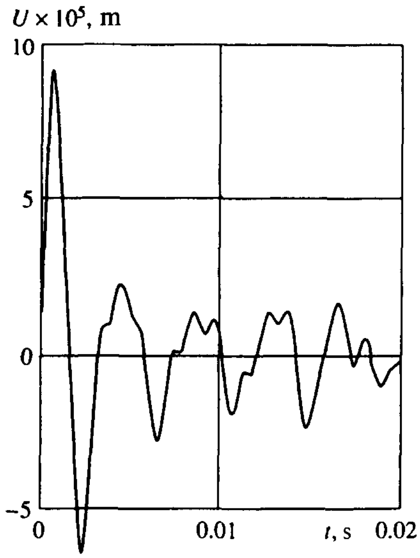


Fig. 3

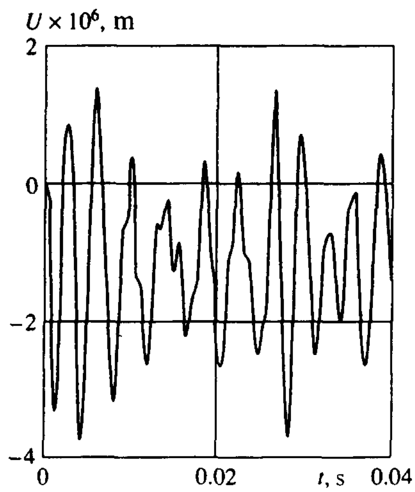


Fig. 4

Figure 2 shows the APFCs plotted according to the system of equations (16) and formula (17), which are practically identical. The average error of the approximation amounts to less than 2%.

Then, using the data given above, we plot the transient by means of the formula

$$u(x,t)=\sum_j \frac{2k_j}{\xi_j}\exp(-\eta_j t)\sin(\zeta_j t) \tag{18}$$

$$\xi_j=(4T_{2j}^2-T_{1j}^2)^{1/2},\quad \eta_j=\frac{T_{1j}}{2T_{2j}^2},\quad \zeta_j=\frac{(4T_{2j}^2-T_{1j}^2)^{1/2}}{2T_{2j}^2}$$

The response of the system to a single impulse, according to formula (18), is shown in Fig. 3. The transient for a stepped force with zero initial conditions is plotted from the formula

$$u_1(x,t)=\sum_j k_j\left\{1-\exp(-\eta_j t)\left(\cos(\zeta_j t)+\frac{T_{1j}}{\xi_j}\sin(\zeta_j t)\right)\right\} \tag{19}$$

and is shown in Fig. 4.

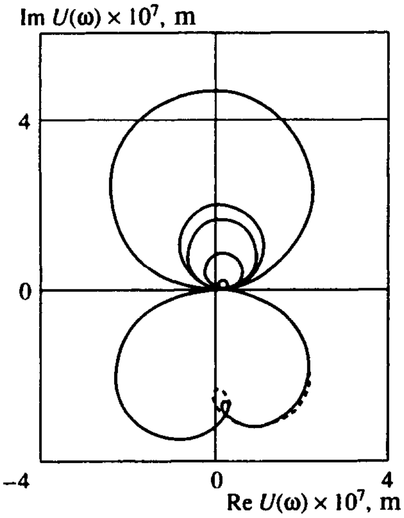


Fig. 5

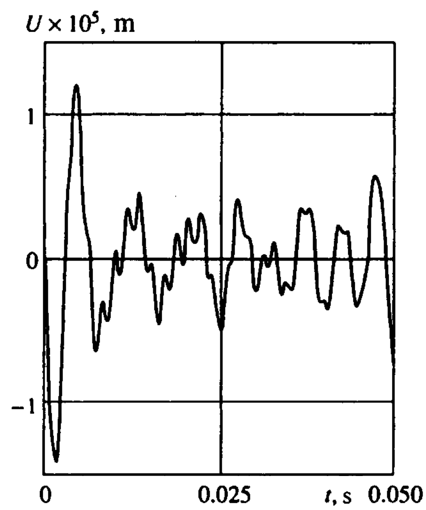


Fig. 6

The second problem is to investigate the dynamics of a conical shell when it collides with a rigid obstacle at $\theta = 135°$; the remaining parameters of the problem are the same as in the previous problem. The velocity of collision is 0.1 m/s. The edge of the shell in contact with the obstacle is rigidly clamped. The perturbing factor in the given case will be the quantity $q = -V_0R$, where $R$ is the mass per unit area of the middle surface. We plot the APFCs, from which a mathematical model of the shell of revolution when it collides with a rigid obstacle is formed. Here, as in the previous case, the average error of the approximation amounted to less than 2%. In the given case, seven terms of series (17) are significant, the coefficients of which are given below

| $j$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| $-k_j \times 10^{12}$, $m^3/N$ | 46340 | 1968 | 1609 | 849 | 161 | 2607 | 2406 |
| $T_{1j} \times 10^9$, s | 12280 | 11020 | 9802 | 8686 | 7305 | 2618 | 2194 |
| $T_{2j}^2 \times 10^7$, $s^2$ | 12400 | 11080 | 9816 | 8818 | 7747 | 2811 | 27820 |

Figure 5 shows the APFCs of the shell (the continuous curves) and of the model (the dashed curve). The result of calculating the response of the system to an impulse is given in Fig. 6.

The third problem differs from the second only in that, after collision, the shell rebounds from the obstacle. Having plotted the APFCs, we set up a mathematical model of the shell of revolution when

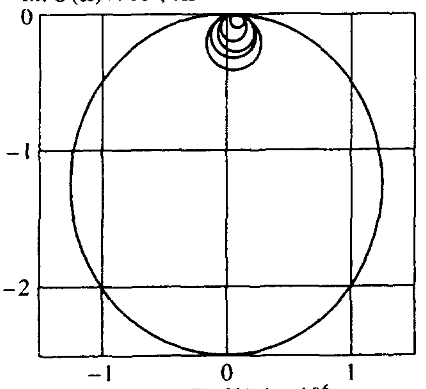


Fig. 7

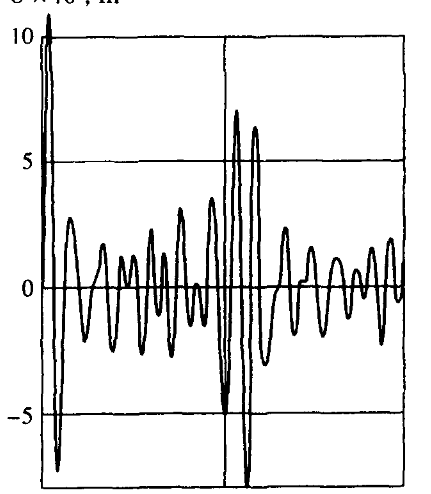


Fig. 8

it collides with a rigid obstacle. The error of the approximation is 2%. Here, six terms of the series (17) are significant, and their coefficients are given below

| $j$ | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| $k_j \times 10^{12}$, m$^3$/N | 29070 | 4242 | 3255 | 1840 | 445 | 24570 |
| $T_{1j} \times 10^9$, s | 11620 | 10170 | 9293 | 7793 | 6874 | 6499 |
| $T_{2j}^2 \times 10^7$, s$^2$ | 11640 | 10310 | 9203 | 8351 | 7577 | 6591 |

Figure 7 shows the APFCs of the shell and of the model, which are practically identical. The result of calculating the response of the system to an impulse is given in Fig. 8.

From the transients it is possible to find the maximum amplitude $A_{max}$, and the static amplitude $A_{st}$, from which, using the formula $K^* = A_{max}/A_{st}$, we find the transmission coefficient. Then, finding the static stress, $\sigma_{st}$, by means of the formula $K^*\sigma_{st} = \sigma_{max}$ we have the maximum stress in the cross-section considered. The given solutions hold for any energy dissipation, including the case when $\gamma = 0$, when semiharmonic undamped vibrations are produced and the solution holds for any time intervals, i.e. for hundreds or thousands of cycles. Like the previous method, the procedure described enables us to investigate the vibrations of shells with any sudden jumps in pressure and to investigate the acoustic vibrations of the shell.

When considering shells of revolution of variable curvature, when $r_1 = r_1(\theta)$, such a shell can be approximated using conical finite elements with different angles $\theta$.

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*Translated by* P.S.C.